

# Projective product coverings and sequential motion planning algorithms in real projective spaces

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## Abstract

For positive integers  $m$  and  $s$ , let  $\mathbf{m}_s$  stand for the  $s$ -th tuple  $(m, \dots, m)$ . We show that, for large enough  $s$ , the higher topological complexity  $\mathrm{TC}_s$  of an even dimensional real projective space  $\mathbb{R}P^m$  is characterized as the smallest positive integer  $k = k(m, s)$  for which there is a  $(\mathbb{Z}_2)^{s-1}$ -equivariant map from Davis' projective product space  $P_{\mathbf{m}_s}$  to the  $(k+1)$ -th join-power  $((\mathbb{Z}_2)^{s-1})^{*(k+1)}$ . This is a (partial) generalization of Farber-Tabachnikov-Yuzvinsky's work relating  $\mathrm{TC}_2$  to the immersion dimension of real projective spaces. In addition, we compute the exact value of  $\mathrm{TC}_s(\mathbb{R}P^m)$  for  $m$  even and  $s$  large enough.

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## 1 Introduction

Michael Farber's notion of topological complexity (TC) was introduced in [5, 6] as a way to study the motion planning problem in robotics from a topological perspective. Due to its homotopy invariance, the concept quickly captured the attention of algebraic topologists who began to study the homotopy TC-phenomenology. In particular, Farber's TC was soon identified as a special instance of a slightly more general concept: Rudyak's higher topological complexity  $\mathrm{TC}_s$ , which recovers Farber's TC if  $s = 2$ , can be thought of as a measure of the robustness to noise of motion planning algorithms in automated multitasking processes ([2, 10]).

Soon after their introduction, the TC-ideas found a highly surprising connection with one of the central problems in last century's main homotopy developments. Namely, it is shown in [7] that, for the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ ,  $\mathrm{TC}_2(\mathbb{R}P^m)$  agrees with  $\mathrm{Imm}(\mathbb{R}P^m)$ , the Euclidean immersion dimension of  $\mathbb{R}P^m$ , provided  $m \neq 1, 3, 7$ . Using the main result in [1], this means that, without restriction on  $m$ ,  $\mathrm{TC}_2(\mathbb{R}P^m)$  can be described, in purely homotopic terms, as the minimal positive integer  $a(m)$ , also denoted by  $\mathrm{axial}(\mathbb{R}P^m)$ , for which the restriction to  $\mathbb{R}P^m \times \mathbb{R}P^m$  of the Hopf multiplication

$$\mu: \mathbb{R}P^\infty \times \mathbb{R}P^\infty \rightarrow \mathbb{R}P^\infty$$

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can be compressed to a map  $\mathbb{RP}^m \times \mathbb{RP}^m \rightarrow \mathbb{RP}^{a(m)}$  —a so called (optimal) axial map. With this in mind, it is natural to ask for the (geometric and homotopic) properties of  $\mathbb{RP}^m$  encoded by the higher analogues  $\mathrm{TC}_s(\mathbb{RP}^m)$ . Such a task is addressed in this paper and, in doing so, we are naturally lead to Davis’ projective product space  $\mathbf{P}_{\mathbf{m}_s}$ , introduced in [4], and defined as the orbit space of  $(S^m)^{\times s}$  by the diagonal (antipodal)  $\mathbb{Z}_2$ -action —in Davis’ notation,  $\mathbf{m}_s$  stands for the  $s$ -tuple  $(m, \dots, m)$ .

In slightly more detail, for  $s \geq 2$ , a natural generalization of the construction in [7, (4.2)] leads to

$$(1) \quad \mathrm{TC}_s(\mathbb{RP}^m) \geq \mathrm{secat}(\pi_s),$$

where  $\pi_s: \mathbf{P}_{\mathbf{m}_s} \rightarrow (\mathbb{RP}^m)^{\times s}$  is the “pivoted axial”  $(\mathbb{Z}_2)^{\times(s-1)}$ -principal bundle whose projection map is induced by the  $s$ -fold cartesian power of the Hopf double cover  $S^m \rightarrow \mathbb{RP}^m$  (further details of this construction are reviewed in the next section). The central result in [7] asserts that (1) is an equality for  $s = 2$ . The proof of such a fact is achieved by

- (I) connecting  $\mathrm{secat}(\pi_2)$  to the existence of (optimal) axial maps  $\mathbb{RP}^m \times \mathbb{RP}^m \rightarrow \mathbb{RP}^{\mathrm{secat}(\pi_2)}$ , and then
- (II) showing how (optimal) motion planners for  $\mathbb{RP}^m$  are encoded by such axial maps.

It is not difficult to prove the right generalization of (I) for  $s \geq 3$  (see Proposition 2.3 below). On the other hand, when  $m$  is even, the validness of a suitable statement generalizing (II) is hinted both by Proposition 3.4 below and by the cohomological calculations in Section 4. In particular, for  $m$  even and  $s$  large enough, we prove that equality holds in (1), and compute the resulting explicit value of  $\mathrm{TC}_s(\mathbb{RP}^m)$  —see Corollary 4.8 below.

On the basis of our results, we conjecture that equality always holds in (1). This would yield a full generalization of Farber-Tabachnikov-Yuzvinsky’s result to the higher TC realm. Proving equality in (1) seems to be inherently more complex when  $s \geq 3$ . See Remarks 3.5–3.7 for a discussion of why proving equality in (1) is elementary for  $s = 2$ , while the corresponding task for  $s \geq 3$  becomes interestingly more intricate.

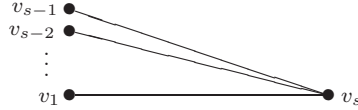
## 2 The projective product covering

For an integer  $s \geq 2$ , the  $s$ -th higher topological complexity of a path connected space  $X$ ,  $\mathrm{TC}_s(X)$ , is defined in [10] as the reduced Schwarz genus of the fibration

$$e_s = e_s^X : X^{[0,1]} \rightarrow X^s, \quad e_s(\gamma) = \left( \gamma\left(\frac{0}{s-1}\right), \gamma\left(\frac{1}{s-1}\right), \dots, \gamma\left(\frac{s-1}{s-1}\right) \right).$$

Thus  $\mathrm{TC}_s(X) + 1$  is the smallest cardinality of open covers  $\{U_i\}_i$  of  $X^s$  so that  $e_s$  admits a continuous section  $\sigma_i$  on each  $U_i$ . The open sets  $U_i$  of such an open cover are called  $s$ -local domains, the corresponding sections  $\sigma_i$  are called  $s$ -local rules, and the resulting family of pairs  $\{(U_i, \sigma_i)\}$  is called an  $s$ -motion planning algorithm for  $X$ . We say that such a family is an optimal  $s$ -motion planning algorithm if it has  $\mathrm{TC}_s(X) + 1$   $s$ -local domains. These ideas are a generalization of the concept of topological complexity introduced by Farber in [5] as a model to study the continuity instabilities in the motion planning of an autonomous system (robot) whose space of configurations is  $X$ . The term “higher” comes by considering the base space  $X^s$  of  $e_s$  as a series of prescribed stages in the robot motion planning, while Farber’s original case  $s = 2$  deals only with the space  $X \times X$  of initial-final stages.

**Remark 2.1.** As shown in [2, pages 2106–2107],  $\text{TC}_s(X)$  can equivalently be defined as the genus of the evaluation map  $X^{\Gamma_s} \rightarrow X^s$ ,  $\gamma \mapsto (\gamma(v_1), \dots, \gamma(v_s))$ , where  $\Gamma_s$  is (the underlying topological space of) a given connected graph, and  $v_1, \dots, v_s$  are  $s$  distinct vertices of  $\Gamma_s$ . In the final section of this paper it will be convenient to take  $\Gamma_s$  to be the graph with exactly  $s$  vertices  $v_1, v_2, \dots, v_s$ , and  $s - 1$  edges  $(v_1, v_s), (v_2, v_s), \dots, (v_{s-1}, v_s)$  depicted as follows:



Most of the existing methods to estimate the higher topological complexity of a space are cohomological in nature. One of the most successful such methods is a special case of Proposition 2.2 below, which is easily proved on the lines of [11, Theorem 4 in page 73].

**Proposition 2.2.** *Let  $h^*$  be a generalized cohomology theory with products. The sectional category of a fibration  $\pi: E \rightarrow B$  is no less than the cup length of elements in the kernel of  $\pi^*: h^*(B) \rightarrow h^*(E)$ .*

Here “cup-length” refers to the maximal number of elements in the indicated ideal having a non-vanishing product.

Later in the paper we will apply Proposition 2.2 to the  $(\mathbb{Z}_2)^{s-1}$ -covering space  $\pi_s$  in (1). The covering space is explicitly defined and studied in this section. Let the group  $(\mathbb{Z}_2)^{s-1}$ , with obvious generators  $\sigma_i$  ( $1 \leq i \leq s - 1$ ), act on  $(S^m)^{\times s}$  so that

$$(2) \quad \sigma_i \cdot (x_1, \dots, x_s) = (x_1, \dots, x_{i-1}, -x_i, x_{i+1}, \dots, x_s).$$

Let  $P_{\mathbf{m}_s}$  be the quotient of  $(S^m)^{\times s}$  by the involution  $\delta \cdot (x_1, \dots, x_s) = (-x_1, \dots, -x_s)$ . It is elementary to check that the induced  $(\mathbb{Z}_2)^{s-1}$ -action on  $P_{\mathbf{m}_s}$  is principal and has orbit space  $(\mathbb{R}P^m)^{\times s}$ . This defines the  $(\mathbb{Z}_2)^{s-1}$ -principal bundle  $\pi_s$ .

For a path  $\gamma$  in  $\mathbb{R}P^m$ , pick a lifting  $\tilde{\gamma}$  through the projection  $S^m \rightarrow \mathbb{R}P^m$ , and note that the class of

$$\left( \tilde{\gamma} \left( \frac{0}{s-1} \right), \tilde{\gamma} \left( \frac{1}{s-1} \right), \dots, \tilde{\gamma} \left( \frac{s-1}{s-1} \right) \right)$$

in  $P_{\mathbf{m}_s}$  does not depend on the chosen lifting  $\tilde{\gamma}$ . We get a map  $(\mathbb{R}P^m)^{[0,1]} \rightarrow P_{\mathbf{m}_s}$  fitting in the commutative diagram

$$(3) \quad \begin{array}{ccc} (\mathbb{R}P^m)^{[0,1]} & \xrightarrow{\quad} & P_{\mathbf{m}_s} \\ & \searrow e_s & \swarrow \pi_s \\ & (\mathbb{R}P^m)^{\times s} & \end{array}$$

which readily yields (1).

The homotopy nature of  $\pi_s$  is described through its classifying map as:

**Proposition 2.3.** *For  $1 \leq i \leq s$  let  $p_i: (\mathbb{RP}^m)^{\times s} \rightarrow \mathbb{RP}^m$  be the  $i$ -th projection,  $\xi_m \rightarrow \mathbb{RP}^m$  be the Hopf bundle over  $\mathbb{RP}^m$ , and  $\mu_s: (\mathbb{RP}^m)^{\times s} \rightarrow (\mathbb{RP}^\infty)^{\times(s-1)}$  classify  $\pi_s$ . Then, for  $1 \leq i \leq s-1$ , the  $i$ -th component  $\mu_{i,s}$  of  $\mu_s$  classifies  $p_i^*(\xi_m) \otimes p_s^*(\xi_m)$ .*

The conclusion of Proposition 2.3 can of course be stated by saying that  $\mu_{i,s}$  is homotopic to the composition of the projection  $p_{i,s}: (\mathbb{RP}^m)^{\times s} \rightarrow \mathbb{RP}^m \times \mathbb{RP}^m$  onto the  $(i, s)$  coordinates, the inclusion  $\mathbb{RP}^m \times \mathbb{RP}^m \hookrightarrow \mathbb{RP}^\infty \times \mathbb{RP}^\infty$ , and the Hopf multiplication  $\mu: \mathbb{RP}^\infty \times \mathbb{RP}^\infty \rightarrow \mathbb{RP}^\infty$ .

*Proof of Proposition 2.3.* Recall  $\delta$  stands for the involution  $(x_1, \dots, x_s) \mapsto (-x_1, \dots, -x_s)$  in  $(S^m)^{\times s}$  so, by definition, the corresponding orbit space is  $P_{\mathbf{m}_s}$ . The total space  $Z_i$  of the  $\mathbb{Z}_2$ -principal bundle classified by the  $i$ -th component  $\mu_{i,s}$  is the quotient of  $(S^m)^{\times s}$  by the actions of  $\delta$  and of those  $\sigma_\ell$  ( $1 \leq \ell \leq s-1$ ) with  $\ell \neq i$ , and where the  $\mathbb{Z}_2$ -principal action on  $Z_i$  is induced by change of signs on the  $i$ -th coordinate. Let  $\lambda_{i,j} \rightarrow \mathbb{RP}^m$  stand for the restriction to the  $j$ -th axis of the latter double covering (axes are taken with respect to the base point in  $\mathbb{RP}^m$  given by the class of  $1 := (1, 0, \dots, 0) \in S^m$ ).

**Case  $j = s$ :** Note that a class in  $\lambda_{i,s}$  has a unique representative of the form  $(1, \dots, 1, x_s)$  and, in these terms, the  $\mathbb{Z}_2$ -principal action on  $\lambda_{i,s}$  is given by

$$\begin{aligned} [(1, \dots, 1, x_s)] &\mapsto [(1, \dots, 1, \overset{i}{-1}, 1, \dots, 1, x_s)] = [(-1, \dots, -1, \overset{i}{1}, -1, \dots, -1, -x_s)] \\ &= [(1, \dots, 1, \overset{i}{1}, 1, \dots, 1, -x_s)], \end{aligned}$$

where the notation  $\overset{b}{a}$  indicates that the number  $a$  appears in the  $b$ -th coordinate of the  $s$ -tuple. Consequently,  $\lambda_{i,s} \rightarrow \mathbb{RP}^m$  is homeomorphic to the Hopf projection  $S^m \rightarrow \mathbb{RP}^m$ .

**Case  $j = i$ :** As above, a class in  $\lambda_{i,i}$  has a unique representative of the form  $(1, \dots, 1, \overset{i}{x_i}, 1, \dots, 1)$  and, now, the  $\mathbb{Z}_2$ -principal action on  $\lambda_{i,i}$  is antipodal on  $x_i$  on the nose. Thus  $\lambda_{i,i} \rightarrow \mathbb{RP}^m$  is also homeomorphic to the Hopf projection  $S^m \rightarrow \mathbb{RP}^m$ .

**Case  $j \notin \{i, s\}$ :** Classes in  $\lambda_{i,j}$  are represented by elements  $(\pm 1, \dots, \pm 1, \overset{j}{x_j}, \pm 1, \dots, \pm 1, \overset{i}{\pm 1}, \pm 1, \dots, \pm 1)$  where, to fix ideas, we have assumed  $j < i < s$  —the case  $i < j < s$  works just as well. Dividing out first by the action of  $\delta$  and of the  $\sigma_\ell$  with  $\ell \notin \{i, j\}$  (and then by the action of  $\sigma_j$ ), we see that  $\lambda_{i,j}$  is given as the quotient of  $S^m \times \mathbb{Z}_2$  by the antipodal action on the first coordinate and with  $\mathbb{Z}_2$ -principal action coming from the antipodal action on the second coordinate. In other words,  $\lambda_{i,j} \rightarrow \mathbb{RP}^m$  is the trivial  $\mathbb{Z}_2$ -bundle.

The conclusion follows.  $\square$

### 3 Motion planning algorithms via equivariant maps

Recall that the  $(k+1)$ -iterated self join-power of a topological space  $X$ ,  $J_k(X)$ , is defined inductively by  $J_k(X) := J_{k-1}(X) * X$  ( $k \geq 1$ ) where  $J_0(X) = X$ . Then, for a topological group  $G$ ,  $B_k G := J_k(G)/G$  is the  $k$ -th stage in Milnor's construction of the classifying space  $BG := J_\infty(G)/G$ , where  $G$  acts diagonally on the vertices of  $J_\infty(G) := \bigcup_{k \geq 0} J_k(G)$  —so barycentric coordinates are preserved.

In what follows  $G_s$  stands for the (discrete) group  $(\mathbb{Z}_2)^{\times(s-1)}$ . By [11, Theorem 9 in page 86], the classifying homotopy class  $\mu_s$  in Proposition 2.3 has a representative factoring in the form

$$(4) \quad (\mathbb{RP}^m)^{\times s} \xrightarrow{\beta_s} B_{\text{secat}(\pi_s)}(G_s) \hookrightarrow B(G_s) \simeq (\mathbb{RP}^\infty)^{\times(s-1)},$$

where  $\beta_s$  is covered by a  $G_s$ -equivariant map  $\alpha_s: P_{\mathbf{m}_s} \rightarrow J_{\text{secat}(\pi_s)}(G_s)$ . Then, in terms of the  $G_s$ -action defined in (2), the composition of the canonical projection  $(S^m)^{\times s} \rightarrow P_{\mathbf{m}_s}$  with  $\alpha_s$  yields a  $G_s$ -equivariant map  $\phi_s: (S^m)^{\times s} \rightarrow J_{\text{secat}(\pi_s)}(G_s)$  satisfying the condition

$$(5) \quad \phi_s(x_1, \dots, x_{s-1}, -x_s) = \sigma_1 \cdots \sigma_{s-1} \cdot \phi_s(x_1, \dots, x_{s-1}, x_s), \text{ for all } (x_1, \dots, x_s) \in (S^m)^{\times s}.$$

**Conjecture 3.1.** *An  $s$ -motion planning algorithm for  $\mathbb{R}P^m$  with  $\text{secat}(\pi_s) + 1$   $s$ -local rules can be constructed out of a map  $\phi_s$  as above. Consequently  $\text{secat}(\pi_s) \geq \text{TC}_s(\mathbb{R}P^m)$ , and (1) becomes an equality for any  $s \geq 2$ .*

The conjecture is motivated in part by (the proof of) [7, Proposition 6.3], which asserts that the case  $s = 2$  of Conjecture 3.1 holds true —see Propositions 3.4 and Remark 3.5 below. Corollary 4.8 in the next section is meant to gather further evidence for the plausibility of Conjecture 3.1. A few additional instances where Conjecture 3.1 holds true are included in the final section of this paper.

**Remark 3.2.** One of our main interests in Conjecture 3.1 is the possibility of obtaining upper bounds for  $\text{TC}_s(\mathbb{R}P^m)$  from the construction of  $G_s$ -equivariant maps  $\phi_s: (S^m)^{\times s} \rightarrow J_k(G_s)$  satisfying (5). Indeed, such a map covers a map  $\beta_s$  as in (4), so that [11, Theorem 9 in page 86] implies  $k \geq \text{secat}(\pi_s)$ , and so  $k \geq \text{TC}_s(\mathbb{R}P^m)$  if Conjecture 3.1 were to hold.

Given spaces  $X$  and  $Y$ , consider the open subspace  $U \subset X * Y$  consisting of the barycentric expressions  $t_0x + t_1y$  with  $(x \in X, y \in Y, 0 \leq t_i, t_0 + t_1 = 1, \text{ and } t_1 > 0)$ . Observe that, if  $Y$  is discrete,  $U$  is a topological disjoint union of open cones with base  $X$  (the cones are open in the sense that they are missing their base). In such terms, the following auxiliary result becomes self-evident.

**Lemma 3.3.** *For  $k \geq 0$ ,  $s \geq 2$ , and  $0 \leq j \leq k$ , consider the open set  $U_j \subset J_k(G_s)$  consisting of the barycentric expressions  $\sum_{\ell=0}^k t_\ell g_\ell$  with  $t_j > 0$  (here, as usual,  $g_\ell \in G_s$ ,  $t_\ell \geq 0$ , and  $\sum t_\ell = 1$ ). Then  $U_j$  is closed under the action of  $G_s$ , and has  $2^{s-1}$  connected components, each of which is open in  $U_j$  and contractible (in itself). Further, the induced  $G_s$ -action on the set of connected components of  $U_j$  has a single orbit.*

**Proposition 3.4.** *Let  $D_s = \{(x_1, \dots, x_s) \in (S^m)^{\times s} : x_i = x_s \text{ for some } i \in \{1, \dots, s-1\}\}$ . The conclusions in Conjecture 3.1 hold true if one starts with a  $G_s$ -equivariant map  $\phi_s: (S^m)^{\times s} \rightarrow J_{\text{secat}(\pi_s)}(G_s)$  satisfying (5) together with one of the following conditions:*

- (1) *For every  $j \in \{0, 1, \dots, \text{secat}(\pi_s)\}$ ,  $\phi_s(D_s)$  intersects at most a single component of  $U_j$ .*
- (2) *For some  $j_0 \in \{0, 1, \dots, \text{secat}(\pi_s)\}$ ,  $\phi_s(D_s)$  is fully contained in some component of  $U_{j_0}$ .*

**Remark 3.5.** The easy fact that, for  $s = 2$ , there exist maps  $\phi_2$  as that assumed in Proposition 3.4 was first noted in [7, Lemmas 5.3 and 5.7]. Explicitly, it is standard that the case  $m = 1, 3, 7$  can be accounted by using the multiplication in the complex, quaternion, and octonion numbers, respectively. For  $m \neq 1, 3, 7$ , since the diagonal inclusion  $\mathbb{R}P^m \hookrightarrow \mathbb{R}P^m \times \mathbb{R}P^m$  is a cofibration, any axial map  $\alpha: \mathbb{R}P^m \times \mathbb{R}P^m \rightarrow \mathbb{R}P^{\text{secat}(\pi_2)}$ , being nullhomotopic on the diagonal<sup>1</sup>, is homotopic to a map  $\alpha': \mathbb{R}P^m \times \mathbb{R}P^m \rightarrow \mathbb{R}P^{\text{secat}(\pi_2)}$  which is (necessarily axial and) actually constant on the diagonal. Then any map  $\phi_2: S^m \times S^m \rightarrow J_{\text{secat}(\pi_2)}(\mathbb{Z}_2) = S^{\text{secat}(\pi_2)}$  covering  $\alpha'$  is a fortiori constant on the diagonal. In particular, such maps  $\phi_2$  satisfy both conditions (1) and (2) in Proposition 3.4 for, obviously, the singleton  $\phi_2(D_2)$  is fully contained in some component of each  $U_j$  satisfying  $\phi_2(D_2) \cap U_j \neq \emptyset$ .

<sup>1</sup>This uses the fact that  $\text{secat}(\pi_2) > m$ , which in turn comes from the assumption  $m \neq 1, 3, 7$  (compare to Remark 3.6).

*Proof of Proposition 3.4.* For  $0 \leq j \leq \text{secat}(\pi_s)$ , set  $V_j = \phi_s^{-1}(U_j) \subseteq (S^m)^{\times s}$ , and  $W_j = q(V_j) \subseteq (\mathbb{RP}^m)^{\times s}$  where  $q$  stands for the composition  $(S^m)^{\times s} \rightarrow \mathbf{P}_{\mathbf{m}_s} \rightarrow (\mathbb{RP}^m)^s$  of canonical projections. Note that the equality

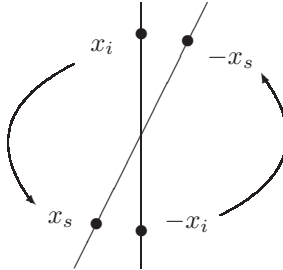
$$(6) \quad V_j = q^{-1}(W_j)$$

holds since  $V_j$  is closed under the action of  $\delta$  and of the  $\sigma_\ell$  with  $1 \leq \ell \leq s-1$  (as the  $G_s$ -equivariant map  $\phi_s$  satisfies (5)). Further, the sets  $W_0, \dots, W_{\text{secat}(\pi_s)}$  form an open cover of  $(\mathbb{RP}^m)^{\times s}$ .

If condition (1) in the statement of the proposition holds, we complete the proof by constructing local sections  $\varsigma_j: W_j \rightarrow (\mathbb{RP}^m)^{\Gamma_s}$  ( $0 \leq j \leq \text{secat}(\pi_s)$ ) for the evaluation map  $(\mathbb{RP}^m)^{\Gamma_s} \rightarrow (\mathbb{RP}^m)^{\times s}$  described at the end of Remark 2.1. Details follow. Fix  $j \in \{0, 1, \dots, \text{secat}(\pi_s)\}$ . If  $\phi_s(D_s)$  intersects  $U_j$ , let  $U_{j,0}$  denote the component of  $U_j$  containing  $\phi_s(D_s) \cap U_j$ ; otherwise, choose any component  $U_{j,0}$  of  $U_j$ . For  $(L_1, \dots, L_s) \in W_j$ , the  $2^s$  elements in  $q^{-1}\{(L_1, \dots, L_s)\}$  lie in  $V_j$ , in view of (6). Also, in view of (5) and the final assertion in Lemma 3.3, exactly two elements in  $q^{-1}(L_1, \dots, L_s)$  have  $\phi_s$ -image in  $U_{j,0}$ . Indeed, if one of the latter elements is  $(x_1, x_2, \dots, x_s)$ , then the other is  $(-x_1, -x_2, \dots, -x_s)$ . Furthermore, in these conditions,

$$(7) \quad \text{if } L_i = L_s \text{ for some } 1 \leq i < s, \text{ then in fact } x_i = x_s,$$

by construction (and in view of Lemma 3.3). Then  $\varsigma_j(L_1, \dots, L_s): \Gamma_s \rightarrow \mathbb{RP}^m$  is defined to be the map whose restriction to the (oriented) edge  $(v_i, v_s)$  describes the uniform-speed motion in  $\mathbb{RP}^m$  from  $L_i$  to  $L_s$  obtained by rotating  $L_i$  toward  $L_s$  along the plane generated by these two lines (no motion if  $L_i = L_s$ ), and in such a way that the corresponding rotation from  $x_i$  to  $x_s$  is performed through an angle smaller than 180 degrees. As shown in the picture below, the latter requirement holds independently of whether one uses  $(x_1, \dots, x_s)$  or  $(-x_1, \dots, -x_s)$ , so that  $\varsigma_j(L_1, \dots, L_s)$  is well defined.



The resulting function  $\varsigma_j$  is clearly a section on  $W_j$  for the evaluation map  $(\mathbb{RP}^m)^{\Gamma_s} \rightarrow (\mathbb{RP}^m)^s$ . Lastly, the continuity of  $\varsigma_j$  follows from (7), and from the facts that  $U_{j,0}$  is open, that  $\phi_s$  is continuous, and that  $q$  is a covering projection.

A minor modification of the above construction is needed in order to complete the proof when condition (2) in the statement of the proposition holds. Indeed, in the notation above, the problematic  $q(D_s)$  is contained in  $W'_{j_0} := W_{j_0}$ , while condition (2) assures that the construction above yields the needed local section  $\varsigma'_{j_0} = \varsigma_{j_0}: W'_{j_0} \rightarrow (\mathbb{RP}^m)^{\Gamma_s}$ . For all other  $j \neq j_0$  we set  $W'_j := W_j - q(D_s)$  (so that the sets  $W'_i$  with  $0 \leq i \leq \text{secat}(\pi_s)$  cover  $(\mathbb{RP}^m)^{\times s}$ ), which (is open and) vacuously avoids the possibility of the failure of (7), thus yielding an obviously continuous local section  $\varsigma'_j: W'_j \rightarrow (\mathbb{RP}^m)^{\Gamma_s}$ .  $\square$



Regarding a potential proof of Conjecture 3.1, the authors believe that, for general  $s \geq 2$ , Proposition 2.3 would have to play a key role in proving the existence of a map  $\phi_s$  as the one assumed in Proposition 3.4. However, the problem seems to be much more subtle for  $s \geq 3$  than the rather straightforward instance  $s = 2$ . We close this section by pinpointing some of the intricacies that are inherent to a potential proof of Conjecture 3.1 via Proposition 2.3 when  $s \geq 3$ , and how this leads to a couple of interesting new challenges in the field (which we hope to address elsewhere).

**Remark 3.6.** We start by discussing the relevance of the inequality

$$(8) \quad \text{secat}(\pi_s: P_{\mathbf{m}_s} \rightarrow (\mathbb{RP}^m)^{\times s}) \geq (s-1)m, \text{ with strict inequality if } m+1 \text{ is not a power of } 2$$

(obtained in (9) and Remark 4.4 below from Proposition 2.2) in a potential proof of Conjecture 3.1. Recall that the isomorphism class of the  $G_s$ -principal bundle  $\pi_s$  has been described in Proposition 2.3 via the homotopy type of its classifying map  $\mu_s: (\mathbb{RP}^m)^s \rightarrow (\mathbb{RP}^\infty)^{\times(s-1)}$ . Of course, the homotopy type of any map  $\beta_s: (\mathbb{RP}^m)^s \rightarrow B_{\text{secat}(\pi_s)}(G_s)$  fitting in the factorization (4) does not have to be determined by that of  $\mu_s$ . Nonetheless, as noted in Remark 3.5, a key fact in the proof of the  $s = 2$  case of Conjecture 3.1 is that any such  $\beta_2$  remains being null homotopic on the diagonal when  $m \neq 1, 3, 7$ , as  $\text{secat}(\pi_2) > m$  for those values of  $m$ . (As explained in [7, Lemma 5.4], the latter inequality turns out to be closely related to Adams' solution of the Hopf invariant 1 problem.) Now, for  $s \geq 3$ , the diagonal is replaced by the “pivoted” diagonal  $q(D_s)$  used at the end of the proof of Proposition 3.4. Then, in order to understand the homotopy properties of the restricted  $\beta_s|_{D_s}$  from the corresponding properties of the restricted  $\mu_s|_{D_s}$  (as in the case  $s = 2$ ), we would need to know that  $\dim(D_s)$  is *strictly smaller* than the connectivity of the inclusion  $B_{\text{secat}(\pi_s)}(G_s) \hookrightarrow B_\infty(G_s) \simeq (\mathbb{RP}^\infty)^{\times(s-1)}$ . Such a condition is assured by (8) if  $m+1$  is not a power of 2, as the latter map is a  $\text{secat}(\pi_s)$ -equivalence (its homotopy fiber agrees with that for the (obviously)  $\text{secat}(\pi_s)$ -equivalence  $J_{\text{secat}(\pi_s)} \rightarrow *$ ), while  $D_s$  is a union of subcomplexes of  $(\mathbb{RP}^m)^{\times s}$  each homeomorphic to  $(\mathbb{RP}^m)^{\times(s-1)}$ , so that  $\dim(D_s) = (s-1)m$ . Consequently, the first task to deal with in a proof of Conjecture 3.1 based on Proposition 3.4 is to decide whether (8) can be improved to a strict inequality when  $m+1$  is a power of 2. As indicated in Example 4.7 below, (8) is in fact an equality for  $m = 1, 3, 7$ , in which case (1) is an equality too. Thus, the real initial task is to decide whether (8) actually improves to a strict inequality for  $m = 2^e - 1$  with  $e \geq 4$ —just as in the case  $s = 2$ . A particularly interesting feature of such a challenge is to understand how a potential strict inequality in (8) would fit within (a possibly generalized form of) the Hopf invariant 1 problem.

**Remark 3.7.** In addition to the considerations in Remark 3.6, it should be noted that, unlike the situation for  $s = 2$ , no map  $\beta_s$  as above can be nullhomotopic on  $D_s$  when  $s \geq 3$  for, in fact,  $\mu_s$  evidently fails to be nullhomotopic on  $D_s$ . Consequently, unlike the situation for  $s = 2$  discussed in Remark 3.5, the issue of being able to “fix” a  $G_s$ -equivariant map  $\phi_s$  as in (5) so to satisfy at least one of the two conditions in Proposition 3.4 requires handling non-trivial homotopy information.

## 4 Cohomology estimates

This section is devoted to estimating the sharpness of (1) by means of cohomological methods. In particular, we show equality for all even  $m$  when  $s$  is large enough. Explicitly, an application of Proposition 2.2 to  $e_s$ , which is a fibrational replacement for the diagonal  $\Delta_s: X \hookrightarrow X^{\times s}$ , yields the lower bound

$$\text{TC}_s(X) \geq \text{zcl}_s^{h^*}(X),$$

where  $\text{zcl}_s^{h^*}(X)$  is the  $h^*$ -cup-length of  $s$ -th zero-divisors in  $X$ , i.e. of elements in the kernel of the induced map  $\Delta_s^*: h^*(X^{\times s}) \rightarrow h^*(X)$  (see [2, Definition 3.8]). In this section we show that, when  $X := \mathbb{RP}^m$  and  $h^* := H^*$  is singular cohomology with mod 2 coefficients,  $\text{zcl}_s(\mathbb{RP}^m) := \text{zcl}_s^{H^*}(\mathbb{RP}^m)$  is in fact a lower bound for the right hand-side in (1), which, for  $m$  odd and  $s$  large enough, agrees with the well known upper bound  $sm \geq \text{TC}_s(\mathbb{RP}^m)$  coming from [2, Theorem 3.9].

Recall that  $H^*((\mathbb{RP}^m)^{\times s}) = H^*(\mathbb{RP}^m)^{\otimes s}$  is the  $\mathbb{Z}_2$ -algebra generated by the classes  $x_i = p_i^*(x)$  subject to the relations  $x_i^{m+1} = 0$ ,  $1 \leq i \leq s$ , where  $x \in H^1(\mathbb{RP}^m)$  is the first Stiefel-Whitney class of  $\xi_m$ , and  $p_i$  is defined in Proposition 2.3. We do not stress the dependence of  $x_i$  on  $s$  because, if  $s' > s$  and  $\pi_{s,s'}: (\mathbb{RP}^m)^{\times s'} \rightarrow (\mathbb{RP}^m)^{\times s}$  is the projection onto the first  $s$  coordinates, then we think of the map induced in cohomology by  $\pi_{s,s'}$  as a honest inclusion. The standard (graded) basis of  $H^*((\mathbb{RP}^m)^{\times s})$  consists of all the monomials  $x_1^{a_1} x_2^{a_2} \cdots x_s^{a_s}$  with  $0 \leq a_i \leq m$ . Note that each  $x_i + x_s$  ( $1 \leq i \leq s-1$ ) is an  $s$ -th zero-divisor, so it pulls back trivially under the evaluation map  $e_s^*$ . In fact:

**Proposition 4.1.** *For  $1 \leq i \leq s-1$ ,  $x_i + x_s$  pulls back trivially under the map  $\pi_s$  on the right of (3).*

*Proof.* The projection  $(S^m)^{\times s} \rightarrow S^m \times S^m$  onto the  $(i, s)$  axes induces a map from  $\pi_s$  to  $\pi_2$  lying over  $p_{i,s}$ . The conclusion then follows since  $x \otimes 1 + 1 \otimes x \in H^1(\mathbb{RP}^m \times \mathbb{RP}^m)$ , the mod 2 Euler class of the exterior product  $\xi_m \otimes \xi_m$ , vanishes under  $\pi_2$ , which is the sphere bundle of  $\xi_m \otimes \xi_m$ .  $\square$

**Lemma 4.2.** *The ideal of  $s$ -th zero-divisors in  $H^*(\mathbb{RP}^m)^{\otimes s}$  is generated by the elements  $x_i + x_s$  in Proposition 4.1.*

*Proof.* Let  $\sum_{(a_1, \dots, a_s)} x_1^{a_1} \cdots x_s^{a_s}$  be the expression of an homogeneous  $s$ -th zero-divisor  $z$  in terms of the standard basis. Note that the number of summands must be even if  $\deg(z) \leq m$ . Thus, it suffices to prove that the following elements lie in the ideal  $I_s$  generated by the binomials  $x_i + x_s$ :

- (i) The sum of any two basis elements in degree at most  $m$ .
- (ii) A basis element in degree greater than  $m$ .

Elements in (i) are easily dealt with by induction on the degree and on the number of common factors. For instance

$$x_1 x_2 + x_3 x_4 = (x_1 x_2 + x_2 x_3) + (x_2 x_3 + x_3 x_4) = x_2(x_1 + x_3) + x_3(x_2 + x_4).$$

Elements in (ii) are dealt with also by an inductive argument based on the fact that, for  $i < j$ ,

$$x_i^{a_i} x_{i+1}^{a_{i+1}} \cdots x_j^{a_j} = (x_i + x_j) \cdot x_i^{a_i-1} x_{i+1}^{a_{i+1}} \cdots x_j^{a_j} + x_i^{a_i-1} x_{i+1}^{a_{i+1}} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j+1} \equiv x_i^{a_i-1} x_{i+1}^{a_{i+1}} \cdots x_{j-1}^{a_{j-1}} x_j^{a_j+1},$$

where the congruence holds module  $I_s$ .  $\square$

Thus (1) extends to

$$(9) \quad sm \geq \text{TC}_s(\mathbb{RP}^m) \geq \text{secat}(\pi_s) \geq \text{zcl}_s(\mathbb{RP}^m).$$

Set  $G(m, s) = sm - \text{zcl}_s(\mathbb{RP}^m)$ , so equality holds in (1) whenever  $G(m, s) = 0$ .

**Lemma 4.3.**  $G(m, 2) \geq G(m, 3) \geq G(m, 4) \geq \cdots \geq 0$ .



*Proof.* Thinking in terms of the expression of elements as sums of the standard basis of  $H^*(\mathbb{RP}^m)^{\otimes s}$ , we see that if  $z \in H^*((\mathbb{RP}^m)^{\times s})$  is a non-zero product of  $s$ -th zero-divisors, then

$$z \cdot (x_1 + x_{s+1})^m = z \cdot (x_{s+1}^m + \cdots) \neq 0.$$

So,  $\text{zcl}_{s+1}(\mathbb{RP}^m) \geq \text{zcl}_s(\mathbb{RP}^m) + m$  and the result follows.  $\square$

**Remark 4.4.** It is elementary to check that  $\text{zcl}_2(\mathbb{RP}^m) = 2^{z(m)} - 1$ , where  $z(m)$  is the integral part of  $\log_2(2m)$  (c.f. [7, Theorem 4.5]). The last line in the previous proof then implies  $\text{zcl}_s(\mathbb{RP}^m) \geq (s-1)m$  with strict inequality if  $m+1$  is not a power of 2. The proof of Theorem 4.5 below is based on a streamlined version of the previous inequality.

The monotonic sequence of non-negative integers in Lemma 4.3 stabilizes, and we denote by  $G(m)$  the corresponding stable value.

**Theorem 4.5.** Assume  $m \equiv 2^e - 1 \pmod{2^{e+1}}$  with  $e \geq 0$ . In other words,  $e$  is the length of the block of consecutive ones ending the binary expansion of  $m$ . For instance,  $e = 0$  if and only if  $m$  is even. Then  $G(m) \leq 2^e - 1$  with equality if  $m$  is even, or if  $m = 2^e - 1$ . In fact,  $G(m, s) \leq 2^e - 1$  for  $s \geq \max\{(m+1)/2^e, 2\}$ . Specifically, if  $m > 2^e - 1$  and  $\sigma$  stands for  $(m+1)/2^e$  (so  $\sigma$  is an integer greater than 2), then the product of  $\sigma$ -th zero-divisors

$$(x_1 + x_\sigma)^{m+2^e} \cdots (x_{\sigma-1} + x_\sigma)^{m+2^e} \in H^*((\mathbb{RP}^m)^\sigma)$$

is non-zero.

**Conjecture 4.6.** In Theorem 4.5, the equality<sup>2</sup>  $G(m) = 2^e - 1$  holds without restriction on  $e$ .

**Example 4.7.** For  $e \geq 1$  and  $s \geq 2$ ,

$$(10) \quad 0 \neq x_1^{2^e-1} x_2^{2^e-1} \cdots x_{s-1}^{2^e-1} + \cdots = (x_1 + x_s)^{2^e-1} (x_2 + x_s)^{2^e-1} \cdots (x_{s-1} + x_s)^{2^e-1} \in H^*((\mathbb{RP}^{2^e-1})^{\times s}),$$

which yields  $G(2^e - 1, s) \leq 2^e - 1$ . The latter inequality is in fact an equality in view of Lemma 4.2 and the fact that the  $2^e$ -th power of any element in  $H^*((\mathbb{RP}^{2^e-1})^{\otimes s})$  vanishes. In the case of the three Hopf spaces  $\mathbb{RP}^1$ ,  $\mathbb{RP}^3$ , and  $\mathbb{RP}^7$ , the  $\text{TC}_s(\mathbb{RP}^m)$ -contents of the assertion  $G(2^e - 1, s) = 2^e - 1$  is strengthened by [9, Theorem 1]:  $\text{TC}_s(\mathbb{RP}^m) = m(s-1)$  for all  $s$  and  $m \in \{1, 3, 7\}$ . Thus, for all  $s \geq 2$  and  $m \in \{1, 3, 7\}$ , the last three terms in (9) are all equal to  $m(s-1)$ .

*Proof of Theorem 4.5.* The case  $m = 2^e - 1$  is accounted for in Example 4.7, so we assume  $m > 2^e - 1$ . The hypothesis on  $m$  and  $e$  implies that the binomial coefficient  $\binom{m+2^e}{2^e}$  is odd, so

$$(x_i + x_\sigma)^{m+2^e} = x_i^m x_\sigma^{2^e} + \text{terms involving powers } x_i^j \text{ with } j < m$$

for  $2 \leq i \leq \sigma$ . Therefore, ignoring basis elements  $x_1^{a_1} \cdots x_\sigma^{a_\sigma}$  having  $a_i < m$  for some  $i \in \{1, \dots, \sigma-1\}$ , the product of  $\sigma$ -th zero-divisors under consideration becomes

$$(x_1^m x_\sigma^{2^e})(x_2^m x_\sigma^{2^e}) \cdots (x_{\sigma-1}^m x_\sigma^{2^e}) = x_1^m x_2^m \cdots x_{\sigma-1}^m x_\sigma^{(\sigma-1)2^e},$$

which is an element of the standard basis.  $\square$

<sup>2</sup>Conjecture 4.6 has recently been proved in [3, Theorem 3.3].

Corollary 4.8 below, a direct consequence of Theorem 4.5, should be compared with the final assertion in Example 4.7.

**Corollary 4.8.** *If  $m$  is even and  $s > m$ , all inequalities in (9) are in fact equalities.*

The hypothesis  $s > m$  can substantially be relaxed in many cases. For instance, [8, Theorem 1.2] implies that the conclusion in Corollary 4.8 remains true for all  $s \geq 3$  if  $m$  is a 2-power. Other concrete instances follow from Propositions 4.2, 4.7 and 4.9–4.12 in [3].

## 5 Examples with $\mathrm{TC}_s(\mathbb{RP}^m) = \mathrm{secat}(\pi_s: \mathbf{P}_{\mathbf{m}_s} \rightarrow (\mathbb{RP}^m)^{\times s})$

In this brief closing section we summarize our knowledge of examples where (1) is either an equality, or holds within one from being so. On the other hand, we are not aware of any case where (1) actually fails to be an equality.

Since  $\mathrm{TC}_s(\mathbb{RP}^1) = s-1$  ([2, Corollary 3.12]), (8) and (9) force (1) to be an equality for  $m = 1$ . In slightly more general terms, and as indicated in Example 4.7, equality in (1) holds for  $m \in \{1, 3, 7\}$ . It would be interesting to give an explicit construction of the corresponding (forced)  $G_s$ -maps  $\phi_s: (S^m)^{\times s} \rightarrow J_{s-1}(G_s)$  satisfying (5). For instance, when  $s = 2$  and  $m = 1$ , so that  $J_{s-1}(G_s) = S^1$ , the required map  $\phi_2$  can be defined by multiplication of complex numbers.

In the previous section we have discussed how Theorem 4.5 provides instances with equality in (1) when  $m$  is even. We now remark that the same arguments show that, in any case, (1) fails from being an equality by at most a unit provided  $m \equiv 1 \pmod{4}$  and  $s \geq \frac{m+1}{2}$  (as in the case of  $m$  even, the restriction imposed by the last inequality can usually be relax substantially).

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